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A STOCHASTIC PROGRAMMING APPROACH TO

WEAPONS INVENTORY PLANNING

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A STOCHASTIC PROGRAMMING APPROACH TO

WEAPONS INVENTORY PLANNING

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CHAPTER I

INTRODUCTION

Implementation of United States defense policy is largely formed around a number of hypothetical situations, or contingencies, that might possibly exist in the future. Some of these contingencies deal with the circumstances under which the U. S. or its allies (or both) might engage in a "hot" war with some adversary. The plan of action for the U. S. and its allies in meeting such situations is called the "contingency plan" and involves not only governmental inter-relationships, but also programs for the actual deployment and utilization of forces. These contingency plans provide the basis for fiscal defense appropriations and allocation of funds within the services.

Once a year defense spending is reviewed by Congress and a budget is forecast for the next five-year period. The budget review inevitably causes a flurry of activity among the services for each to "get its piece of the cake." Contrary to popular belief, however, a great portion of the recommendations of the services is backed by good systems analysis. One of these studies, The Non-Nuclear Consumables Manual, generated by Headquarters, Air Force, was the origin of this thesis problem.

The Non-Nuclear Consumables Manual provides the basis each year for the Air Force (USAF) recommended operating budget for conventional (non-nuclear)weaponry and ancillary equipment. To compile this manual, which is ultimately incorporated in the Joint Chief of Staff's War Mobilization Plan, various contingencies are analyzed and interpreted as to the USAF role in each. For each situation involving conventional warfare, the contingency plans specify time limits within which the contingency objectives must be accomplished. USAF intelligence experts decide what types of targets must be destroyed by the Air Force tactical commands, and how many of each type must be put out of operation in the given time limit if the objectives are to be satisfied. At this point it is generally left to the operations/systems analysts to decide what is needed to accomplish these tasks cheaply, and within operational limitations.

Unfortunately, the method for performing the latter effort is extremely cumbersome and somewhat inaccurate. A thorough study of the techniques presently being used has pointed out some of the following rather glaring deficiencies:

1) Given a fixed aircraft force size, the present procedure attempts to address the problem from the standpoint of maximizing the number of targets destroyed, while minimizing total inventory cost.

2) Weapon delivery tactics are preplanned and the inventory fitted to them. Both aircraft survivability and weapons effectiveness vary greatly with delivery tactics, thus implying the cost to achieve a given level of target destruction changes accordingly. The present approach fixes, arbitrarily, one of the more important decision variables open to the planner.

3) Variation of aircraft attrition to enemy defenses with delivery conditions is not reflected at all in the planning process. Further,

attrition rates are not considered in the availability of combat aircraft in the later stages of a conflict.

4) Weather is considered incorrectly. At present the probability that a target is destroyed by an air delivered munition in perfect weather is degraded by the probability that the weather meets the minimum standards for that weapon used. The degraded probability is then used directly as representative of the capabilities of the weapon system.

The objective of this thesis is to design an analytical framework within which the non-nuclear consumables problem may be worked. This procedure will be one which will correct most of the above shortcomings and provide more insight into the problem of proposing a minimum cost inventory of conventional weaponry which accomplishes the objectives of the contingency plans.

CHAPTER II

PROBLEM FORMULATION

The solution procedure for any problem must be built around the available data. To begin this effort it will be assumed that military planners have categorized various potential targets and estimated how many of each must be destroyed to accomplish the ambitions of the pertinent contingency plans. Logically speaking, this is necessary in that it is merely a statement of the ultimate objectives of the problem. Practically, however, this is probably the facet of the problem which is most open to challenge, both militarily and politically. It stands to reason, of course, that any worthwhile analysis would include a sensitivity study of this area.

There are many factors that characterize targets, the most important of which is their physical characteristics. For example, one would expect a 1000 pound bomb to have a decidedly different probability of destroying a truck than it would a heavily armored tank. Thus categorization of targets into specific types, based on their physical characteristics, is necessary. Of course no two individual targets are exactly alike, but it is reasonable to assume that a boundary may be drawn so that the differences in effects of a particular weapon type on targets within a category are negligible. (This has, in fact been done by a large joint service group, as will be discussed later.)

There are other things that cause differentiation among targets other than the physical characteristics. One of these is the range from the target to the nearest USAF base. The farther the attacking aircraft has to fly to get to the target, the greater the cost of the attack. Thus, a categorization by range is necessary, and expressible in terms of the number of targets of a particular type within a specified range band from the nearest USAF base.

Due to aircraft attrition to enemy defenses, it is generally much more expensive to attack heavily defended targets than lightly defended ones. Here also a categorization of targets by the level of their defense is necessary. An additional implication of this is that if a particular type of target is characteristically heavily defended (e.g. enemy airfields), and exacts a heavy toll of aircraft lost, then the planner may elect to design an inventory that necessitates no greater than a given number of missions be flown against that target type, thus implying an operational constraint imposed by the target itself.

Weather is another problem in attacking targets and is very restrictive in certain areas. The planner may also categorize weather, as to its goodness or badness, in terms of the ceiling and visibility limitations. For example, it might be that, based on cloud ceiling and visibility, the weather situation has been divided into categories of good, fair and bad weather. This division adds flexibility to the formulation in a number of ways. Some targets, such as bridges, if not of immediate tactical importance, will not move and pilots may wait until good weather to attack. Others, such as tanks, are always of immediate importance, and must be hit regardless of weather. Thus, one might see few or no bridge targets in the poor weather band and tanks in all weather bands. Likewise, truck move mainly at night so that might exist only in a poor weather band.

In defining a weapon, first and of foremost importance are the physical characteristics of the munition package itself: its warhead, glide characteristics (e.g. a free falling bomb as opposed to a boosted rocket), guidance, or lack of it, and so on. These, together with the accuracy with which the munition may be delivered, are the driving factors in determining whether a given weapon will destroy its target.

There are several other important factors to take into consideration in addressing a weapon's capabilities and differentiating between it and other weapons. Certain weapons, for example, are releasable at very low altitudes, and consequently achieve greater accuracy, while others are greatly restricted as to where they may be released. For instance, a 1000 pound bomb, released by a pilot flying level at 200 feet, will be observed to detonate approximately under his aircraft, no matter how fast he is going. This is an example of an undesirable delivery condition. However, if the bomb is equipped with a more expensive, high drag tailfin, then it may be delivered at that altitude. Since both weapon accuracy (and therefore effectiveness) and the risk of losing the aircraft to enemy defenses vary with delivery condition, with no loss in generality, the same munition delivered from two different conditions may be called two different weapons.

Some munitions may only be delivered from certain types of aircraft and, on these aircraft, the weapon must be loaded a special way. The type of aircraft used, of course, affects the total cost of using the munition. Additionally the method of loading the weapon on the aircraft, as well as how many are loaded, affects the mission success probability.

If the contingency plan implies that hostilities will exist over an extended period of time, the total number of missions that may be flown by USAF aircraft will probably be large. However, due to the fact that some munitions may require special ancillary equipment and/or may only be flown on certain types of aircraft the total number of missions that can be flown with a particular weapon is generally much less than the total number of missions that may be flown by all aircraft.

To formalize the definitions of "weapon" and "target" let the following apply:

target - defined to be a piece or pieces of equipment or structure which must be destroyed in the process of meeting contingency plans. Targets are categorized and differentiated as to type by physical characteristics, range to nearest USAF base, defensive posture, and the weather situation existing when the target is attacked.

weapon - the combination of munition hardware (warhead, guidance package, etc.), type of aircraft and its loadout, and delivery condition. It will be assumed that a weapon with a given set of the above features will be further characterized by the target it is used against. Thus, no "weapon" will be used against two different types of targets.

The above definition of "weapon" seems a bit cumbersome at first; however it will add greatly to the simplicity of mathematical proofs later. In the end, the optimal inventory may be obtained by simply adding all weapons in the optimal solution that have a common munition.

Finally, cost must be considered. Herein only the unit cost of the weapon is addressed, though later it will be obvious that a problem with fixed charges may be worked. The objective of this problem is to minimize the total expected cost of the inventory subject to certain constraints. Associated with each munition is a unit cost which is generally well known. But there are other costs that are not so obvious. One is the cost to fly the aircraft to the target and return. Another is the cost of replacing aircraft that are lost in combat. Since both defense level and range to target are included in the description of the target, to the cost of the munition may be added an aircraft operating cost, as well as an expected attrition cost. The latter is the cost of the aircraft (and pilot if applicable) multiplied by the probability that the delivery aircraft is lost to hostile action.

To give the reader an idea of the order of magnitude of the numbers of targets and weapons that will be considered in applications of the model developed herein suppose defense levels, ranges to targets, and weather situations are divided into three categories each. There will be around thirty categories of targets by their physical characteristics alone. Together this implies that about 8100 types of targets will be attacked. Likewise, with weapons, 30 types of munitions with three delivery conditions and five aircraft types each yields 240 weapons. Needless to say, some orderly procedure is necessary to handle a problem of this magnitude.

To summarize the previous discussion, in Chapter I it was stated that a need exists to revamp the methods of analysis leading to USAF specification of required munition inventory levels. In Chapter II the concepts of "weapon," "target," "mission," and "cost" as they apply to the problem have been defined. Generally, the problem may be defined as that of finding a set of weapons that will destroy the required number of targets of each type, subject to operational constraints. Further, this set of weapons must be such that there is no other set that will accomplish the objectives, within the constraints, with a lower total expected cost. The following chapters will be devoted to making this problem mathematically precise and showing how it may be solved.

CHAPTER III

DERIVATION OF PROBLEM CONSTRAINTS

The problem to this point has been loosely stated as that of minimizing a linear expected cost function subject to certain contingency requirements being met. The objective now is to put this into a more exact mathematical format.

Target Constraints

The discussion of Chapters I and II has inferred that the military planner should stockpile enough weapons hardware to "assure" destruction of a rather large fraction of the adversary's military potential. What this fraction actually is will not be discussed here though the following development will indicate that it may be treated as being uncertain. In practice the analyst will probably parameterize on this quantity and attempt to locate a point of diminishing returns with respect to the size of the inventory.

The act of delivering a weapon on a target does not insure the target's destruction by any means. In fact, one is often surprised how low the kill probability (p_k) is for certain weapons, even against the targets for which they were designed. A large joint service group has been in existence for several years with the primary task of quantifying all the factors involved with estimating conventional weapon's effectiveness against certain targets. Such estimates of weapon p_k's are available to the military planner and will be used in the application of this thesis to inventory planning.

Success on any one mission is an event with a certain (known) probability implying that the number of targets destroyed (successes) out of any given number of missions is actually a random variable. This fact, along with fixing the number of targets of each type that must be destroyed, enables the planner to state his objectives in more precise terms. Call the number of targets of the i^{th} type that must be destroyed H_i, and assume the planner is also willing to state a minimum acceptable level of assurance, say α_i, where $0 < \alpha_i < 1$, that the inventory will do the job required. Thus if N_i is the number of targets of the i^{th} type destroyed, a probability statement, which the inventory must satisfy, has been implied:

$$P_r[N_i \geq H_i] \geq \alpha_i \tag{1}$$

To facilitate solution of the problem, (1) must be put into more manageable form. Let p_{ij} be the probability that a weapon of the j^{th} type will destroy the i^{th} target, and let x_{ij} be the number of these weapons used. (This is synonymous with saying x_{ij} missions with the j^{th} type of weapon are flown against the i^{th} type of target.) Then the number of targets of the i^{th} type that are destroyed is a binomially distributed random variable (assuming the missions are independent in their effectiveness), n_{ij}, with mean $x_{ij}p_{ij}$, and variance $x_{ij}p_{ij}(1 - p_{ij})$. For planning purposes it is reasonable to assume that attacks on individual targets are independent, so the number of targets of the i^{th} type destroyed by all the inventory weapons is a random variable also, the sum of the realizations of the n_{ij}:

$$N_i = \sum_{j=1}^{W_i} n_{ij}$$

where W_i is the number of weapons considered for use against the i^{th} target.

Note that n_{ij} may be thought of as the number of successes in x_{ij} trials, when the probability of success on one trial is p_{ij}. If x_{ij} is "large," then a common approach in applied probability theory is to approximate the distribution of n_{ij} with a normal distribution. That is, for large x_{ij}, $(n_{ij} - x_{ij}p_{ij})/\sqrt{x_{ij}p_{ij}(1 - p_{ij})}$ approximately has the distribution N(0,1). Since N_i is the sum of independent random variables which are approximately normally distributed, then the distribution for N_i may be approximated by the normal distribution with mean μ_i and variance σ_i^2, where:

$$\mu_i = \sum_{j=1}^{W_i} x_{ij}p_{ij} = \sum_{j=1}^{W_i} x_{ij}m_{ij}$$

$$\sigma_i^2 = \sum_{j=1}^{W_i} x_{ij}p_{ij}(1 - p_{ij}) = \sum_{j=1}^{W_i} x_{ij}v_{ij}$$

letting, for notational simplicity, $v_i = p_{ij}(1 - p_{ij})$ and $m_{ij} = p_{ij}$.

Consider the inequality in the probability statement (1),

$$N_i \geq H_i$$

Subtracting the mean of N_i from both sides and dividing by the standard deviation,

$$\frac{N_i - \mu_i}{\sigma_i} \geq \frac{H_i - \mu_i}{\sigma_i}$$

and the probability statement becomes:

$$P_r\left[\frac{N_i-\mu_i}{\sigma_i} \geq \frac{H_i-\mu_i}{\sigma_i}\right] \geq \alpha_i$$

Now $(N_i - \mu_i)/\sigma_i$ is approximately distributed N(0,1) and $(H_i - \mu_i)/\sigma_i$ is a constant. Let t_{α_i} be such that integration of the standard normal probability density function between t_{α_i} and $+\infty$ yields a probability of α_i. Then $(H_i - \mu_i)/\sigma_i \leq t_{\alpha_i}$ is equivalent to $P_r[(N_i - \mu_i)/\sigma_i \geq (H_i - \mu_i)/\sigma_i] \geq \alpha_i$ because $(N_i - \mu_i)/\sigma_i$ has the standard normal distribution. For example, for $\alpha_i = 0.95$, $t_{\alpha_i} = -1.645$ and (1) is equivalent to the restriction $(H_i - \mu_i)/\sigma_i \leq -1.645$.

The above is the certainty equivalent approach to the problem originally formulated by Charnes and Cooper (reference 1). It says that satisfaction of the deterministic "certainty equivalent" statement

$$\frac{H_i-\mu_i}{\sigma_i} = \frac{H_i - \sum_j m_{ij}x_{ij}}{[\sum_j v_{ij}x_{ij}]^{1/2}} \leq t_{\alpha_i} \qquad (2)$$

is probabilistically the same as satisfying (1). Thus, the somewhat nebulous statement in (1) has been transformed to a precise mathematical constraint. The planner selects levels for the x_{ij} that minimize the linear objective function subject to constraints of the form of (2). In the final solution it will take $\sum_{i=1}^{T} x_{ij}$ weapons of type j to satisfy the contingency requirements, where T is the number of types of targets.

The expression in (2) may be put into a form that is easier to handle, but not without some complications. Rearranging the terms (2) becomes

$$\sum_j m_{ij}x_{ij} + t_{\alpha_i} \left(\sum_j v_{ij}x_{ij}\right)^{1/2} - H_i \geq 0 \qquad (3)$$

or in vector form:

$$\underline{m}_i \underline{x}_i + t_{\alpha_i} (\underline{v}_i \underline{x}_i)^{1/2} - H_i \geq 0 \, . \qquad (4)$$

By rearranging and squaring (4), the inequality may be rewritten, though the direction of the inequality sign depends on the sign of t_{α_i} and the relative magnitudes of the absolute values of $(\underline{m}_i \underline{x}_i - H_i)$ and $t_{\alpha_i} (\underline{v}_i \underline{x}_i)^{1/2}$. Performing these operations (4) becomes

$$(\underline{m}_i \underline{x}_i - H_i)^2 \leq t_{\alpha_i}^2 \underline{v}_i \underline{x}_i \quad \text{if} \quad t_{\alpha_i} \quad \underline{\text{and}} \quad |\underline{m}_i \underline{x}_i - H_i| \leq |t_{\alpha_i} (\underline{v}_i \underline{x}_i)^{1/2}| \qquad (5)$$

$$(\underline{m}_i \underline{x}_i - H_i)^2 \geq t_{\alpha_i} \underline{v}_i \underline{x}_i \quad \text{if} \quad t_{\alpha_i} \quad \underline{\text{and}} \quad |\underline{m}_i \underline{x}_i - H_i| \geq |t_{\alpha_i} (v_i x_i)^{1/2}| \qquad (6)$$

and

$$(\underline{m}_i \underline{x}_i - H_i)^2 \leq t_{\alpha_i}^2 \underline{v}_i \underline{x}_i \quad \text{if} \quad t_{\alpha_i} \geq 0 \qquad (7)$$

Equations (5), (6), and (7) all have the unfortunate characteristic that there are two possible vectors $\underline{x}_i$ that solve these expressions at equality. One is such that $(\underline{m}_i \underline{x}_i - H_i) \leq 0$ and the other is such that $(\underline{m}_i \underline{x}_i - H_i) \geq 0$. Which is desired depends on whether t_{α_i} is greater than, or less than zero.

Expanding and collecting terms (5) and (7) become

$$(\underline{m}_i \underline{x}_i)^2 - \underline{a}_i \underline{x}_i + H_i^2 \leq 0 \tag{8}$$

and (6) becomes

$$(\underline{m}_i \underline{x}_i)^2 - \underline{a}_i \underline{x}_i + H_i^2 \geq 0 \tag{9}$$

where the elements of the vector $\underline{a}_i$ are

$$a_{ij} = (2H_i m_{ij} + t_{\alpha_i}^2 v_{ij})$$

Associating a defense level with each target, Department of Defense analysts are now able to estimate, for each weapon, the probability that the delivery aircraft is lost to enemy defenses when attacking that target. Assuming the planner wishes to choose an inventory composition that will insure that no greater than a certain percent of his attack aircraft will be lost he may develop a certainty equivalent similar to the one in (2). A very real fact in all past wars has been that when losses attacking any type of target become too great, missions against that type of target are discontinued. Thus, the far-sighted analyst might reflect this point in an attrition certainty constraint for each type of target.

Linear Constraints

If there were only constraints of the certainty equivalent form, as in (2), then the obvious solution would be to fly as many missions, with the cheapest weapon, as necessary to satisfy each constraint. Unfortunately there is a limit on how many missions can be flown in any finite period of time. This implies an additional constraint of the form

$$\sum_{i=1}^{I} \sum_{j=1}^{W_i} x_{ij} \leq b_m$$

where b_m is the total number of missions available.

An upper bound on the number of missions flown with a certain weapon type might be imposed by equipment limitations or some other cause. A constraint of this form would be written

$$\sum_{i=1}^{1} x_{ij} \leq b_j$$

If the analyst wishes to limit the number of missions flown against some particular type of target, he would impose a constraint of the form

$$\sum_{j=1}^{W_i} x_{ij} \leq b_i$$

Since all the uses of this model cannot be foreseen at this time, the desire is to preserve as much generality in the problem formulation as possible. For this reason no restrictions of any sort will be placed on the linear constraints. It is assumed, however, that the only non-linear constraints are the certainty equivalents.

CHAPTER IV

MATHEMATICAL DEVELOPMENT

In reference 4, M. Resh has addressed a problem similar in mathematical characteristics to the one attacked here. Resh's problem, one of machine loading with a stochastic constraint matrix, involves the maximization of a linear function subject to a set of concave certainty equivalents. It will be shown that the problem of Chapter III is one of minimizing a linear function subject to a set of linear constraints as well as certainty constraints.

In both cases, since the effect of the non-linear constraints is to make the feasible region non-convex, classical constrained optimization techniques are not applicable and one may resort to linear approximations of the certainty equivalents. The method of development of these linear approximations, and their goodness of fit will be the subject of the next two chapters. Because of the similarities in the two problems, some of Resh's proofs are readily adaptable to the weapon's inventory problem and will be used with a minimum amount of change. Where possible the same notation is used to facilitate easy comparison.

Shape of the Nonlinear Constraints

The weapons inventory problem may be stated as follows:

$$\begin{aligned}
&\text{minimize} \quad \underline{cx} \\
&\text{subject to} \quad L_1\underline{x}_1 + L_2\underline{x}_2 + \ldots\ldots\ldots\ldots + L_T\underline{x}_T \leq \underline{b}_0 \qquad (1) \\
&\qquad\qquad A_1\underline{x}_1 \qquad\qquad\qquad\qquad\qquad \leq \underline{b}_1
\end{aligned}$$

$$
\begin{array}{llll}
g_1(\underline{x}_1) & & & \geq 0 \\
 & A_2\underline{x}_2 & & \leq \underline{b}_2 \\
g_2(\underline{x}_2) & & & \geq 0 \\
 & & A_T\underline{x}_T & \leq b_T \\
 & & g_T(\underline{x}_T) & \geq 0
\end{array}
$$

$$\text{and} \quad x_{ij} \geq 0$$

The A_i matrices are linear and pertain only to the i^{th} target. The L_i are also linear and $g_i(\underline{x}_i)$ is of the form (4), Chapter III. In this and the following sections it will be assumed that $g_i(\underline{x}_i)$ pertains to target certainty equivalents and H_i will be considered as being deterministic. Development of the problem is applicable to other formulations as well.

THEOREM I: $g_i(\underline{x}_i) = \underline{m}_i\underline{x}_i + t_{\alpha_i}(\underline{v}_i\underline{x}_i)^{1/2} - H_i$ is convex for $t_{\alpha_i} \leq 0$ and concave for $t_{\alpha_i} > 0$ over the nonnegative orthant.

PROOF: With no loss in generality the constant and linear terms may be dropped. Noting that a function f is convex if and only if αf is convex for each $\alpha > 0$ and concave for each $\alpha < 0$, it suffices to show that $(\underline{vx})^{1/2}$ is concave. Let $\underline{x}_1$ and $\underline{x}_2$ be arbitrary nonnegative vectors. To prove concavity it must be shown that

$$[\underline{v}(\lambda\underline{x}_1 + (1-\lambda)\underline{x}_2)]^{1/2} \geq \lambda(\underline{v}\,\underline{x}_1)^{1/2} + (1-\lambda)(\underline{v}\,\underline{x}_2)^{1/2} \tag{2}$$

for each $\lambda\varepsilon(0,1)$. Note that

$$[(\underline{v}\,\underline{x}_1)^{1/2} - (\underline{v}\,\underline{x}_2)^{1/2}]^2 \geq 0$$

$$\text{i.e. } \underline{v}\,\underline{x}_1 + \underline{v}\,\underline{x}_2 \geq 2(\underline{v}\,\underline{x}_1)^{1/2}(\underline{v}\,\underline{x}_2)^{1/2} \tag{3}$$

Let $\lambda \varepsilon (0,1)$ and multiply the inequality in (3) by $\lambda(1-\lambda) > 0$. Performing this operation and rearranging terms (3) becomes

$$\underline{v}[\lambda \underline{x}_1 + (1-\lambda)\underline{x}_2] \geq \lambda^2(\underline{v}\,\underline{x}_1) + (1-\lambda)^2(\underline{v}\,\underline{x}_2) + 2\lambda(1-\lambda)(\underline{v}\,\underline{x}_1)^{1/2}(\underline{v}\,\underline{x}_2)^{1/2} =$$

$$[\lambda(\underline{v}\,\underline{x}_1)^{1/2} + (1-\lambda)(\underline{v}\,\underline{x}_2)^{1/2}]^2 \qquad (4)$$

Then (2) follows by taking the square root of both sides of (4).

If $t_{\alpha_i} \geq 0$ for all i, then the problem is one of minimizing a linear function over a convex set bounded by linear and non-linear constraints. There are many algorithms that apply to problems of this form, even when x_{ij} must be integer (for example, see Witzgall, ref. 6). The more interesting case is when $t_{\alpha_i} \leq 0$, and the feasible set becomes a non-convex region. One of the reasons for formulating the inventory problem in this manner is to guarantee, with a high degree of certainty, that the inventory chosen will meet the objectives of the contingency plans. Since $t_{\alpha_i} \leq 0$ implies $\alpha_i \geq 0.5$, this is by far the more interesting case. For these reasons it will be assumed in further development that $\alpha_i \geq 0.5$ and hence $t_{\alpha_i} < 0$ for all i. (If $t_{\alpha_i} = 0$, the certainty constraint becomes linear and the problem may be solved as a linear program.)

Linear Approximations - Lower Bound

Because of the convexity of the $g_i(x_i)$, and the apparent hopelessness of verifying a global optimal solution for the problem in (1), it seems a reasonable approach to attempt to bound the problem with linear constraints, then assess the accuracy of the solution. In this way possibly some sacrifices in feasibility will be made and these will be discussed later.

It has been implied, in previous discussions, that the solution vector, $\underline{x}$, for the problem in (1), may be divided into T column vectors $\underline{x}_i$, $i = 1, \ldots, T$. Each of these $\underline{x}_i$ are associated with a particular target and no two targets have a component of $\underline{x}$ in common. Addressing the i^{th} target, let S_i be the set $S_i = \left\{\underline{x} \mid g_i(\underline{x}_i) \geq 0;\ A_i \underline{x}_i \leq \underline{b}_i;\ x_{ij} \geq 0\right\}$. If S is the feasible region for the problem (1), then S_i contains S. Additionally define $R_i = \left\{\underline{x} \mid g_i(\underline{x}_i) \geq 0;\ x_{ij} \geq 0\right\}$, and note that R_i contains S_i. In bounding the feasible region S with linear constraints, support hyperplanes to the R_i sets will be constructed. It will be shown that these support hyperplanes, together with the linear constraints of (1), define a region that is a good approximation to S. The derivation of these support hyperplanes follows.

Consider the certainty equivalent for the i^{th} target, $g_i(\underline{x}_i)$, at equality, i.e., $g_i(\underline{x}_i) = 0$. (The locus of points satisfying this equation in two-space is depicted in Figure 1.) If all components of $\underline{x}_i$ are zero except the j^{th}, then $g_i(\underline{x}_i) = 0$ is a quadratic equation in x_{ij}. There are two roots to this equation, as discussed previously. If $t_{\alpha_i} < 0$, then $m_{ij}x_{ij}$ must be greater than H_i and the root that satisfies the probability statement (1) of Chapter III, noted as X_{ij}, is

$$X_{ij} = \frac{U_i + \sqrt{U_i^2 + 4m_{ij}H_i}}{2m_{ij}}$$

where $U_i = -t_{\alpha_i}$. Physically, X_{ij} is the minimum number of missions with weapons of the j^{th} type needed to satisfy the certainty constraint relating to the i^{th} target, provided no other weapons are used. In as much as $H_i > 0$, the value of the root X_{ij} is always greater than zero. Alternately considering each of the W_i weapons as above, and computing the respective root, X_{ij}, for each, a set of X_{ij}, $j = 1,2,\ldots,W_i$, is obtained. A hyperplane then may be constructed such that it passes through the points $(X_{i1},0,\ldots,0),(0,X_{i2},0,\ldots,0),\ldots,(0,\ldots,0,X_{iW_i})$.

The constructed hyperplane supports the region R_i and is defined by the set of points satisfying $\underline{h}_i\underline{x}_i = 1$, where $h_{ij} = 1/X_{ij}, j = 1,2,\ldots,W_i$. If $\underline{X}_{ij}$ is defined as $\underline{X}_{ij} = X_{ij}\underline{e}_j$, where $\underline{e}_j = (0,\ldots,0,1,0,\ldots,0)$, with 1 as its j^{th} component. It is seen that $\underline{h}_i\underline{x}_i = 1$ passes through the points $\underline{X}_{ij}$ for $j = 1,\ldots,W_i$. For a pictorial representation of this construction in two-space, refer to Figure 1.

By construction the hyperplane contains points in R_i. A further criteria that must be satisfied if it is to be a support hyperplane is that all $\underline{x} \varepsilon R_i$ lie in one of the half spaces defined by $\underline{h}_i\underline{x}_i = 1$. This is equivalent to stating the following theorem:

THEOREM II: $\underline{x} \varepsilon R_i$ implies that $\underline{h}_i\underline{x}_i \geq 1$.

PROOF: To prove the theorem it will be shown that the set $R_i^* = \{\underline{x} | \underline{h}_i\underline{x}_i \geq 1;\ \underline{x} \geq 0\}$ contains R_i.

Any point, $\underline{y}_i$, on the plane may be expressed as a convex combination of the $\underline{X}_{ij}$ (since it can be shown that these vectors constitute a basis in E^{W_i}). That is,

$$y_i = \sum_{j=1}^{W_i} \lambda_j \underline{X}_{ij}, \quad 0 \leq \lambda_j \leq 1, \quad \sum_{j=1}^{W_i} \lambda_j = 1 \qquad (6)$$

In fact, any $\underline{x}_i = (x_1, x_2, \ldots, x_{W_i})$ lying in the nonnegative orthant may be expressed as a constant multiple of some $\underline{y}_i$ lying on the constructed hyperplane:

$$\underline{x}_i = k\underline{y}_i; \quad \text{where } \underline{x}_i \geq 0,\ \underline{y}_i \geq 0,\ \ k \geq 0, \text{ and } \underline{h}_i\underline{y}_i = 1 .$$

Note by the convexity of g_i if $\underline{y} = \sum_{j=1}^{W_i} \lambda_i \underline{X}_{ij}$, $\lambda_j \epsilon (0,1)$, $\sum_{j=1}^{W_i} \lambda_j = 1$,

$$g_i\left[\sum_{j=1}^{W_i} \lambda_i \underline{X}_{ij}\right] \leq \sum_{j=1}^{W_i} \lambda_j g_i(\underline{X}_j) = 0 \tag{7}$$

where the right side of (7) equals zero by construction. Rewriting this in vector form:

$$g_i(\underline{y}_i) = \underline{m}_i\underline{y}_i + t_{\alpha_i}(\underline{v}_i\underline{y}_i)^{1/2} - H_i \leq 0 \tag{8}$$

Let $\underline{x}_i \,\epsilon\, R_i$ and $\underline{x}_i = k\underline{y}_i$, where $\underline{h}_i\underline{y}_i = 1$ and k is a constant. By (8), letting $t_{\alpha_i} = -U_i$, since it was assumed earlier that $t_{\alpha_i} < 0$,

$$g_i(\underline{y}_i) \leq 0 \quad \text{or} \quad \underline{m}_i\underline{y}_i \leq U_i(\underline{v}_i\underline{y}_i)^{1/2} + H_i \tag{9}$$

But if $\underline{x}_i \,\epsilon\, R_i$, then

$$g_i(\underline{x}_i) = g_i(k\underline{y}_i) \geq 0 \quad \text{or} \quad k\underline{m}_i\underline{y}_i \geq k^{1/2}U_i(\underline{v}_i\underline{y}_i)^{1/2} + H_i \tag{10}$$

By contradiction, assume $k < 1$. Then $k < k^{1/2}$ and

$$k^{1/2}U_i(\underline{v}_i\underline{y}_i)^{1/2} + H_i > kU_i(\underline{v}_i\underline{y}_i)^{1/2} + kH = k\underline{m}_i\underline{y}_i \geq k^{1/2}U_i(\underline{v}_i\underline{y}_i)^{1/2} + H_i$$

which is impossible, so $k \geq 1$. Since $\underline{h}_i\underline{y}_i = 1$, $\underline{h}_i\underline{x}_i = k\underline{h}_i\underline{y}_i \geq 1$. Therefore $\underline{x} \,\epsilon\, R_i^*$.

It has been shown above that every $\underline{x} \varepsilon R_i$ is also contained in R_i^*. It is for this reason that, if $g_i(\underline{x}_i) \geq 0$ is replaced with $\underline{h}_i\underline{x}_i - 1 \geq 0$, and the associated linear programming problem is solved, the optimal solution may be infeasible if it lies on $\underline{h}_i\underline{x}_i = 1$. It will be shown later, however, that this infeasibility is usually very small. The replacement of $g_i(\underline{x}_i) \geq 0$ with $\underline{h}_i\underline{x}_i - 1 \geq 0$ is depicted pictorially in Figure 2.

Linear Approximations - Upper Bound

In the preceding section a linear lower bound for S_i was derived by constructing a supporting hyperplane for the larger set R_i. Replacement of $g_i(\underline{x}_i) \geq 0$ with $\underline{h}_i\underline{x}_i \geq 1$ in (1) results in a linear program which, when solved, may, or may not have a solution feasible to (1). In order to assess this potential infeasibility an upper support hyperplane to each non-linear constraint is derived. This plane, parallel to $\underline{h}_i\underline{x}_i = 1$, will be found by solving the optimization problem

$$\text{maximize } \underline{h}\,\underline{x} \tag{11}$$

$$\text{subject to} \qquad g(\underline{x}) \leq 0; \; \underline{x} \geq 0$$

where the i subscript has been temporarily dropped.

Let $\bar{R}$ be the feasible set for this problem, i.e., $\bar{R} = \left\{\underline{x} \,|\, g(\underline{x}) \leq 0; \; \underline{x} \geq 0\right\}$. Then by the convexity of $g(\underline{x})$, a local optimum is also a global optimum. If (11) has a finite optimum solution, $\hat{\underline{x}}$, then for all $\underline{x}$ in $\bar{R}$, $\underline{h}\,\underline{x} \leq \underline{h}\,\hat{\underline{x}} = \delta$. Thus, $\underline{h}\,\underline{x} = \delta$ contains at least one point of $\bar{R}$ and all points of $\bar{R}$ are in one of the half spaces defined by $\underline{h}\,\underline{x} = \delta$, and so $\underline{h}\,\underline{x} = \delta$ is a support hyperplane for $\bar{R}$.

The problem in (11) has one constraint (in addition to the non-negativity constraints) and is a nonlinear, convex programming problem solvable by any one of a number of well known iterative techniques. Recognition of one of the less obvious characteristics of the weapons inventory problem, however, permits a direct solution by the Kuhn-Tucker conditions. Recalling the definitions of "weapon" and "target," one sees that no two weapons will have exactly the same kill probability against a given target. For example, a single bomb dropped from an aircraft at 3000 feet above the target would be delivered with greater accuracy than the same bome dropped from, say, 3500 feet. Albeit these differences might be small in some cases, nevertheless, they may be assessed and are sufficient to generate the development of Appendix I.

In Appendix I it is shown that (11) must have a solution, and that the solution vector, $\hat{\underline{x}}$ has only two positive components. Noting $(1 - p_j)$ as α_j, the two components are seen to correspond to the maximum and minimum α_j values (or, respectively, the minimum and maximum p_j's) for the weapons used against the target in question. The optimum solution to (11) is

$$\hat{\underline{x}} = (0,\ldots,0,\hat{x}_\ell,0,\ldots,0,\hat{x}_u,0,\ldots,0)$$

where $\hat{x}_u$ corresponds to $\alpha_u = \max_j \alpha_j$, and $\hat{x}_\ell$ to $\alpha_\ell = \min_j \alpha_j$. The realizations of these components are

$$\hat{x}_u = \frac{1}{m_u} \cdot \frac{\hat{V} - \alpha_\ell \hat{M}}{\alpha_u - \alpha_\ell} \tag{12}$$

$$\hat{x}_\ell = \frac{1}{m_\ell} \cdot \frac{\alpha_u \hat{M} - \hat{V}}{\alpha_u - \alpha_\ell} \tag{13}$$

where $\hat{V}$ and $\hat{M}$ are given by

$$\hat{V} = \frac{U^2}{4} \; \frac{\alpha_u y_u - \alpha_\ell y_\ell}{y_u - y_\ell} \tag{14}$$

$$\hat{M} = H + U\hat{V}^{1/2} \tag{15}$$

and y_ℓ and y_u are as defined in Appendix I, equation (A.19).

Since only two components of $\hat{\underline{x}}$ are positive, the right-hand side of the hyperplane, δ, is found to be

$$\delta = \underline{h}\,\underline{x} = h_u x_u + h_\ell x_\ell \; .$$

The point $\hat{\underline{x}}$ has been derived as that which gives a maximum value to $\underline{h}\,\underline{x}$ over the set $\bar{R}$. If $\underline{x}$ is a point such that $\underline{x} \geq 0$, and $\underline{h}\,\underline{x} > \delta$, then $\underline{x}$ cannot be in $\bar{R}$ and must be in $T = \{\underline{x} | g(\underline{x}) > 0; \; \underline{x} \geq 0\}$. (Including the boundary, T is seen to be the same set as R.) It follows, then, that any point, $\underline{x} \geq 0$, satisfying $\underline{h}\,\underline{x} \geq \delta$ must also satisfy the certainty constraint $g(\underline{x}) \geq 0$.

Replacement of $g_i(\underline{x}_i) \geq 0$ with $\underline{h}_i \underline{x}_i \geq \delta$ results in another linear programming problem referred to in the following chapters as the "upper bound" problem. By the above argument it is clear that the upper bound problem solution is always feasible to the original nonlinear problem. For a pictorial representation of the upper bound problem in two-space, see Figure 2.

Bounds on the True Optimal Solution

Applying the techniques of previous sections, two hyperplanes may be derived for each of the probability constraints. The non-linear constraint lies completely between these two planes in the feasible region. It should be remarked again that, though these two hyperplanes are support hyperplanes for the regions R and $\bar{R}$, they do not necessarily constitute support planes for the original problem in (1).

Let S_U be the feasible set for the problem in (1) where the nonlinear constraints have been replaced by the upper bound hyperplanes $\underline{h}_i\underline{x}_i \geq \delta$. Similarly, let S_L be the feasible set replacing the non-linear constraints with the lower bound hyperplanes $\underline{h}_i\underline{x}_i \geq 1$. If S is the feasible set for the original problem it is seen that $S_U \subset S \subset S_L$. If $\underline{cx}_U^*$ and $\underline{cx}_L^*$ are the optimal solutions to the upper and lower bound problems respectively, then it is obvious that

$$cx_U^* \geq \underline{cx}^* \geq \underline{cx}_L^*$$

Unfortunately, though the optimal value of the objective function is bounded as above, similar bounds cannot be placed on the components of $\underline{x}^*$. This is because one is not able to tell which constraints will be binding in the final solution to the original problem without actually solving that problem.

It is possible, however, to use $\underline{x}_U^*$ and $\underline{x}_L^*$ to derive another solution which is a feasible improvement on $\underline{x}_U^*$. Note that, in the constraints that replaced the certainty equivalents are dropped, the remaining linear constraints in each problem are the same. Thus, they define the same set. Letting this set be named P, it is seen that both $\underline{x}_L^*$ and $\underline{x}_U^*$ are necessarily feasible to P. Further, since P is formed by linear constraints, it is convex, and a convex combination of $\underline{x}_L^*$ and $\underline{x}_U^*$ is also in P.

Letting $\underline{x}_L^*$ be of the form $\underline{x}_L^* = (\underline{x}_{1L}^*, \underline{x}_{2L}^*, \ldots, \underline{x}_{WL}^*)$, suppose $\underline{x}_L^*$ is infeasible to one or more of the certainty equivalents. That is, $g_i(\underline{x}_{iL}^*) < 0$ for some i. Recall that $\underline{x}_U^*$ is always feasible. It follows that, by the continuity of the g_i's, there exists some convex combination of $\underline{x}_L^*$ and $\underline{x}_U^*$, say $\underline{x}_c$, that is feasible to all g_i and thus to the entire problem. $\underline{x}_c$ may be expressed as

$$\underline{x}_c = \gamma\ \underline{x}_U^* + (1 - \gamma)\underline{x}_L^*$$

and, if $\underline{x}_L^*$ is infeasible, $0 < \gamma < 1$.

Noting $z_U^* = \underline{c}\underline{x}_U^*$ and $z_L^* = \underline{c}\underline{x}_L^*$, and $z_U^* > z_L^*$ (if $\underline{x}_L^*$ is infeasible),

$$z_c = \underline{c}\underline{x}_c = \gamma z_U^* + (1 - \gamma) z_L^*$$

Since $0 < \gamma < 1$ and $0 < (1 - \gamma) < 1$, $z_L^* < z_c < z_U^*$, and $\underline{x}_c$ is a feasible improvement on $\underline{x}_U^*$. Breaking $\underline{x}_U^*$ up into component vectors like $\underline{x}_L^*$ for the i^{th} certainty equivalent is obtained (using the certainty equivalent form in (9) of Chapter III)

$$\left\{\underline{d}_i[\gamma \underline{x}_{iU}^* + (1 - \gamma)\underline{x}_{iL}^*]\right\}^2 - a_i[\gamma \underline{x}_{iU}^* + (1 - \gamma)\underline{x}_{iL}^*] + b_i \geq 0 \qquad (16)$$

Expanding and collecting terms, (16), at equality, is seen to be a quadratic in γ:

$$\gamma^2\left\{\underline{d}_i[\underline{x}_{iU}^* - \underline{x}_{iL}^*]\right\}^2 + \gamma\left\{2\underline{d}_i\underline{x}_{iL}^*[\underline{d}_i\underline{x}_{iU}^* - 1] + \underline{a}_i[\underline{x}_{iL}^* - \underline{x}_{iU}^*]\right\} \qquad (17)$$
$$- \left\{(\underline{d}_i\underline{x}_{iU})^2 - \underline{a}_i\underline{x}_{iL}^* + b_i\right\} = 0$$

Solving (17) there are two roots, $R_i^{(1)}$ and $R_i^{(2)}$. For one of the roots, say $R_i^{(1)}$, $g_i(\underline{x}_c)$ is greater than, or equal to zero if $\gamma \geq R_i^{(1)}$. Similarly, if $\gamma \leq R_i^{(2)}$, then $g_i(\underline{x}_c) \geq 0$. It should be clear that one of the roots is between zero and one. This is because the continuity of g_i implies the existence of a convex combination of $\underline{x}_U^*$ and $\underline{x}_L^*$ that makes $g_i(\underline{x}) = 0$. Thus, there is a γ such that $R_i^{(1)} \leq \gamma$ or $R_i^{(2)} \geq \gamma$ implies $g_i(\underline{x}_{1c}) \geq 0$.

Performing a similar operation on each certainty equivalent and adding the requirement that $0 \leq \gamma \leq 1$ results in the following one variable optimization problem:

$$
\begin{aligned}
\text{minimize} \quad & \underline{c}[\gamma \underline{x}_U^* + (1-\gamma)\underline{x}_L^*] = \underline{c}\,\underline{x}_L^* + \gamma \underline{c}(\underline{x}_U^* - \underline{x}_L^*) \qquad (18) \\
\text{subject to:} \quad & \text{either} \quad \gamma \geq R_1^{(1)} \quad \text{or} \quad \gamma \leq R_1^{(2)} \\
& \text{either} \quad \gamma \geq R_2^{(1)} \quad \text{or} \quad \gamma \leq R_2^{(2)} \\
& \qquad \vdots \\
& \text{either} \quad \gamma \geq R_T^{(1)} \quad \text{or} \quad \gamma \leq R_T^{(2)} \\
& 0 \leq \gamma \leq 1
\end{aligned}
$$

That (18) must have a solution is again intuitively reasoned by the implied existence of a convex combination of $\underline{x}_L^*$ and $\underline{x}_U^*$ that makes all $g_i(\underline{x}_{1c}) \geq 0$. Derivation of γ yields the sought improvement, $\underline{x}_c$.

Either of the two solutions, $\underline{x}_U^*$ or $\underline{x}_L^*$, offer an easily obtained approximation to $\underline{x}^*$. The worth of these approximations are, of course, a function of their accuracy. The following chapter will be devoted to an exploration of that accuracy as a function of the parameters of the model.

Before proceeding, the reader may wish to solidify ideas presented herein through the use of an example. A numerical example, small by the standards of the problems worked in practice, is given in Appendix II. This problem illustrates some of the concepts addressed in both the previous and subsequent chapters and is provided for the purpose of clarification.

CHAPTER V

SENSITIVITY ANALYSIS OF APPROXIMATIONS

In Chapter IV a method was derived by which the optimum value of the objective function may be bounded through solving two linear programs. The first linear program uses a linear approximation to the certainty constraints and results in a solution which may be slightly infeasible. The optimal solution vector for the second problem, using the linear upper bound, will probably result in "over kill"; that is, having a certainty of the given α that more than the required H targets are killed.

If the solution stops here, the analyst must decide which solution to use as the "optimal" solution. In deciding whether to use one of the L.P. solutions or to attempt some improvement technique, it is best to know how good are the available solutions using the approximations, and what are the maximum errors.

Define the two linear problems as

$$\text{Minimize } \underline{cx} \tag{1}$$

$$\begin{aligned}
\text{Subject to: } & L_1\underline{x}_1 + L_2\underline{x}_2 + \cdots\cdots + L_T\underline{x}_T \le \underline{b}_0 \\
& A_1\underline{x}_1 \le \underline{b}_1 \\
& \underline{h}_1\underline{x}_1 \ge 1 \\
& \quad \cdot \\
& \quad \quad \cdot \\
& \quad \quad \quad \cdot \\
& A_T\underline{x}_T \le b_T \\
& \underline{h}_T\underline{x}_T \ge 1 \\
& x_{ij} \ge 0
\end{aligned}$$

and

$$\text{Minimize } \underline{c}\underline{x} \tag{2}$$

$$\begin{array}{lll} \text{Subject to:} & L_1\underline{x}_1 + L_2\underline{x}_2 + \cdots\cdots\cdots\cdots & L_T\underline{x}_T \leq \underline{b}_0 \\ & A_1\underline{x}_1 & \leq \underline{b}_1 \\ & \underline{h}_1\underline{x}_1 & \geq \delta_1 \\ & \quad \cdot & \\ & \qquad \cdot & \\ & \qquad\quad \cdot & \\ & & A_T\underline{x}_T \leq \underline{b}_T \\ & & \underline{h}_T\underline{x}_T \geq \delta_T \\ & \qquad x_{ij} \geq 0 & \end{array}$$

Problem (1) is the lower bound problem and (2) is the upper bound problem. Let the optimal solution vector to (1) be noted as $\underline{x}_L^*$ and that of problem (2) as $\underline{x}_U^*$. In the following it will be assumed that the analyst has chosen to use the solution vector $\underline{x}_L^*$ as the "optimal" solution to the original problem. This seems most reasonable because it will yield a lower cost than (2) and it is very possible that the requirements of the contingency plans have been somewhat over stated anyway, thus making a small infeasibility safe.

If (1) is solved, errors, or infeasibilities, result from two sources: a) the assumption that the number, n_{ij}, of targets destroyed by any x_{ij} weapons of the j^{th} type is normally distributed (remember it is actually a binomially distributed random variable), and; b) the non-linear certainty constraint has been approximated with a hyperplane.

Addressing a) first, suppose x_{ij} weapons of the j^{th} type were used against the i^{th} target. Then the probability that at least H_{ij}

targets are destroyed by this weapon type is actually

$$P_r(n_{ij} \geq H_{ij}) = 1 - \sum_{k=0}^{H_{ij}-1} \binom{x_{ij}}{k} p_{ij}^k (1 - p_{ij})^{x_{ij}-k} \tag{3}$$

Similar to the development in Chapter III the probability statement that must be satisfied is

$$P_r(n_{ij} \geq H_{ij}) \geq \alpha_i \tag{4}$$

Then, using the normal approximation, a certainty constraint could be derived like the one below:

$$p_{ij}x_{ij} - U[p_{ij}(1 - p_{ij})x_{ij}]^{1/2} - H_{ij} \geq 0 \tag{5}$$

where $-U = t_{\alpha_i}$ (t_{α_i} has been assumed to be negative). Theoretically, (3) could be used in the programming problem, but in practice it would probably be impossible to solve any realistic problem. It is for this reason (5) has been chosen to replace the constraint (4). It is the accuracy of this approximation that must be assessed.

Reference 3 gives tabulated values of the cumulative binomial distribution which may be used to find values of x_{ij} that satisfy (4) for certain H_{ij}, p_{ij}, and α_i. Solution of (5) at equality yields a value for comparison of accuracy. This has been done for α_i of 0.67 and 0.95, H_{ij} of 5, 10, and 50, and p_{ij} of 0.05, 0.1, 0.5 and 0.9, and tabulated in Tables 1 and 2.

In the tables x_{ij}^n represents the solution of (5) using the normal approximation, while x_{ij}^b is the value obtained through the use of the binomial tables (see reference 3). Needless to say, x_{ij}^b gives the

Table 1. Error in Certainty Constraint Approximation for $\alpha = 0.67$

p_{ij}	H_{ij}	x^n_{ij}	x^b_{ij}	% error	p_{ij}	H_{ij}	x^n_{ij}	x^b_{ij}	% error
0.05	5	111	113	-1.8	0.05	10	21	22	0
	10	219	220	-0.4		50	104	105	-0.9
0.10	5	55	56	-1.8	0.90	5	6	6	0
	10	109	100	9.0		10	11	11	0
	50	526	529	-0.6		50	57	56	1.8
0.50	5	11	11	0					

Table 2. Error in Certainty Constraint Approximation for $\alpha = 0.95$

p_{ij}	H_{ij}	x^n_{ij}	x^b_{ij}	% error	p_{ij}	H_{ij}	x^n_{ij}	x^b_{ij}	% error
0.05	5	189	180	5.0	0.50	10	28	28	0
	10	318	312	1.9		50	117	119	-1.7
0.10	5	93	88	5.7	0.90	5	7	7	0
	10	157	155	1.3		10	13	13	0
	50	618	622	-0.6		50	60	60	0

correct solution to (4). (In some cases interpolation of the data from the reference was necessary, resulting in significant percentage errors when x_{ij}^{b} was small.) When $x_{ij}^{n} - x_{ij}^{b}$ is expressed as a percent of the real value, x_{ij}^{b}, as it is in the tables then the error percentage is generally less than five percent and never greater than 10 percent. (x_{ij}^{n} and x_{ij}^{b} were rounded up to the next whole number when they had fractional parts, thus resulting in a large number of zero errors.)

Generally the errors tend to be smaller when x_{ij}^{b} is large. This occurs when either H_{ij} is large and/or p_{ij} is small. This is by far the more common case when a "real life" problem is worked. All in all it should be clear that the approximation is quite good.

Errors also tend to be larger, in the positive direction, for large α_i. This equates to choosing an inventory which has more than the minimum required capability (if the normal approximation is used).

Turning to the error of type b), it has been stated that an $\underline{x}_L^*$ may perhaps be infeasible. Suppose $\underline{x}_L^*$ is infeasible to the ith target's probability constraint. Then accepting x_L^* amounts to a relaxation of the constraint in (1) of Chapter III. That is, $\underline{x}_L^*$ satisfies a probability statement of the type

$$\text{(i)} \quad P_r(N_i \geq H_i) \geq \alpha_i' \quad \text{where } \alpha_i' < \alpha_i$$

$$\text{or} \quad \text{(ii)} \quad P_r(N_i \geq H_i') \geq \alpha_i \quad \text{where } H_i' < H_i$$

Evaluation of the infeasibility in terms of the second statement is probably the more meaningful since it relates directly to a physical quantity, the targets themselves. (ii) says that at $\underline{x}_L^*$ the inventory will destroy H_i' targets at the required certainty level; however, this number is less than desired originally. Note that

$$\underline{m}_i \underline{x}_L^* - u_i[\underline{v}_i \underline{x}_L^*]^{1/2} = H_i' \leq H_i \tag{6}$$

and $H_i - H_i'$ constitutes the amount of infeasibility in $\underline{x}_L^*$.

It would be desirable to find out how great is the maximum infeasibility in the lower bound problem, and relate it to the problem parameters, i.e. p_{ij}, H_i, and α_i. This amounts to solving a problem of the type:

$$\text{Maximize } H_i - [\underline{m}_i \underline{x}_i - U_i(\underline{v}_i \underline{x}_i)^{1/2}] = \text{Maximize } -g_i(\underline{x}_i) \tag{7}$$

$$\text{Subject to: } \underline{h}_i \underline{x}_i \geq 1; \text{ and } x_{ij} \geq 0 .$$

Note that here the objective function relates to the certainty constraints of the original problem. Also note that the concavity of $-g_i(\underline{x}_i)$ insures that the local optima in problem (7) are also global optima.

Temporarily dropping the i subscript, the Kuhn-Tucker conditions for (7) may be written

$$\gamma h_j - \frac{\partial g(\hat{\underline{x}})}{\partial x_j} \leq 0 \qquad j = 1, 2, \ldots \tag{8}$$

$$\hat{x}_j [\gamma h_j - \frac{\partial g(\hat{\underline{x}})}{\partial x_j}] = 0 \quad j = 1, 2, \ldots \tag{9}$$

$$\hat{\underline{x}} \geq 0 \tag{10}$$

$$\gamma(\underline{h}\hat{\underline{x}} - 1) = 0 \tag{11}$$

$$\gamma \geq 0 \tag{12}$$

where $\hat{\hat{\underline{x}}}$ is now the optimum solution to the problem in (7).

The Kuhn-Tucker conditions (8) - (12) are seen to be similar in form to those given for the problem in (11) of Chapter IV (see Appendix I). In fact, it may be seen through a similar analysis that only two components of $\underline{x}$ are positive, and that these correspond to α_u and α_ℓ (as noted previously). Thus, it may be shown that the optimal solution to (7) is of the form

$$\hat{\hat{\underline{x}}} = (0,\ldots,0,\hat{\hat{x}}_u,0,\ldots,0,\hat{\hat{x}}_\ell,0,\ldots,0) \tag{13}$$

Defining $\hat{\hat{V}}^{1/2}$ as $(\underline{v}\hat{\hat{\underline{x}}})^{1/2}$, a procedure like that followed in Appendix I yields

$$\hat{\hat{V}}^{1/2} = \frac{U}{2}\,\frac{y_u\alpha_u - y_\ell\alpha_\ell}{y_u - y_\ell} \tag{14}$$

Equation (13) and the definition of $\hat{\hat{V}}^{1/2}$ indicate that

$$\underline{v}\hat{\hat{\underline{x}}} = v_u\hat{\hat{x}}_u + v_\ell\hat{\hat{x}}_\ell = \frac{U^2}{4}\left(\frac{y_u\alpha_u - y_\ell\alpha_\ell}{y_u - y_\ell}\right)^2 \tag{15}$$

Two equations are needed to solve for $\hat{\hat{x}}_u$ and $\hat{\hat{x}}_\ell$. One is provided by (15) and the other is obtained by realizing that

$$\gamma = \frac{y_u y_\ell(\alpha_u - \alpha_\ell)}{y_u\alpha_u - y_\ell\alpha_\ell} \tag{16}$$

Thus, because of (11), $\underline{h}\hat{\hat{\underline{x}}} = 1$, or

$$\underline{h}\hat{\hat{\underline{x}}} = h_u\hat{\hat{x}}_u + h_\ell\hat{\hat{x}}_\ell = 1 \tag{17}$$

Solving (15) and (17) simultaneously, it is found that

$$\hat{\hat{x}}_u = \frac{h_\ell \hat{\hat{V}} - v_\ell}{v_u h_\ell - v_\ell h_u} \tag{18}$$

and

$$\hat{\hat{x}}_\ell = \frac{v_u - \hat{\hat{V}} h_u}{v_u h_\ell - v_\ell h_u} \tag{19}$$

Using (18) and (19) the maximum infeasibility in the lower bound problem may be assessed. A computer program was written to assess this error for various values of the parameters H_i, p_{ij}, and α_i. The results are shown in Figures 1 - 4. Note that in (18) and (19), given a particular H_i and α_i, the level of infeasibility is determined solely by the $\hat{\hat{x}}_u$ and $\hat{\hat{x}}_\ell$ associated with α_u and α_ℓ. However,

$$\alpha_{ij} = \frac{v_{ij}}{m_{ij}} = \frac{p_{ij}(1 - p_{ij})}{p_{ij}} = (1 - p_{ij}) \tag{20}$$

Consequently, the infeasibility of a certainty constraint, given H_i and α_i, may be assessed in terms of the maximum and minimum p_{ij} of the weapons used in that certainty constraint.

The figures depict the maximum percent infeasibility in the lower bound problem, $(H_i - H_i')/H_i$, as a function of the maximum and minimum p_{ij}.

From the graphs a number of things are apparent. First, the percentage of error increases rapidly as the maximum p_k of the candidate weapons approaches 1.0. For conventional weaponry this is a very uncommon situation, though, and even for a small number of targets the approximation is very good. For example, with $H_i = 10$ and the maximum and minimum p_k's 0.5 and 0.05 respectively, the maximum infeasibility is 1.0 percent for $\alpha = 0.95$ and 0.2 percent for $\alpha = 0.67$. Referring to Tables 1 and 2, for $\alpha = 0.95$ and $H_i = 10$, the normal approximation errors themselves are 0% and 1.9% for $p_k = 0.5$ and $p_k = 0.05$ respectively (for $\alpha = 0.67$ the errors are 0% and -0.4% respectively). Recalling that in the tables the data were rounded off before percentages calculated, one sees the errors are comparable.

The second point to observe is that for a given value of H_i a requirement for less certainty introduces less of an error. This is obvious from the formulation of the certainty constraint in that, as t_α approaches zero, the certainty constraint approaches linearity. Recall, however, that the tables show that greater certainty results in a tendency to a larger error in the positive sense, i.e. an overstocking of weaponry. But the linear approximation in the lower bound problem results in understocking. Thus, in a sense, the errors are somewhat compensating, resulting in a reduction of total error.

Finally, note from the graphs that, as H_i gets larger, the error percentage goes down. Recalling the previous example, at $H_i = 10$, $\alpha_i = 0.95$, with maximum and minimum p_k's of 0.5 and 0.05 the percent infeasibility is 1.0%, or about 0.1 target. In solving the linear program, the number of missions flown would be rounded up to the next higher whole number and this "fraction" of a target would be taken care of. At $H_i = 100$ the error is about 0.2%, or about 0.2 of a target and the same remark applies here.

In conclusion, for most cases, the two approximations used herein are very accurate. Errors only become significant when one of the weapons has an extremely high p_k, say 0.95 or higher. In the field of conventional weapons this is a rare case indeed.

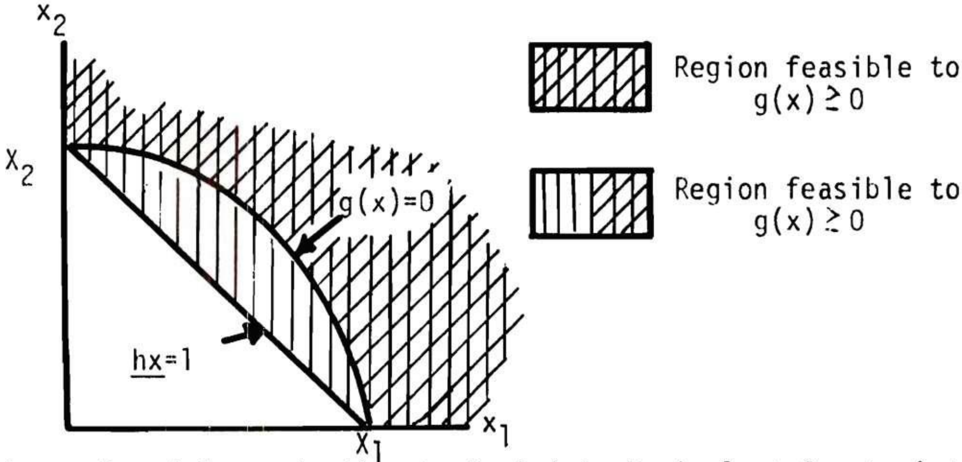

Figure 1. Lower Bound Approximation to Certainty Equivalent Constraint

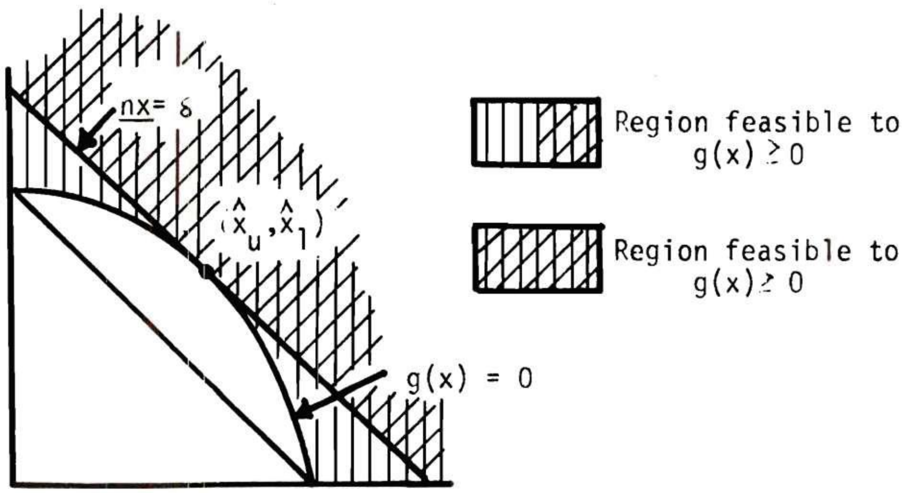

Figure 2. Upper Bound Support Plane

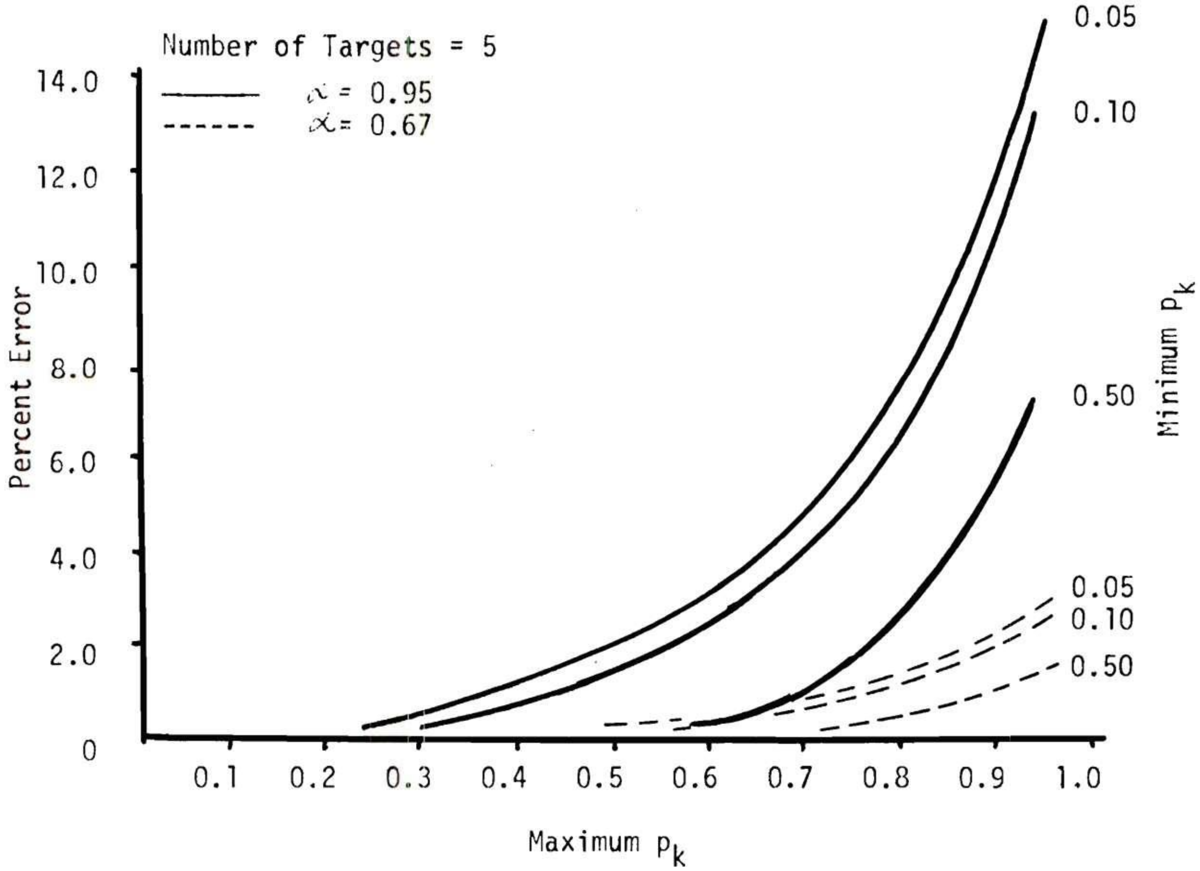

Figure 3. Maximum Infeasibility in Lower Bound Linear Approximation versus Maximum and Minimum p_k for $\alpha = 0.95$ and 0.67.

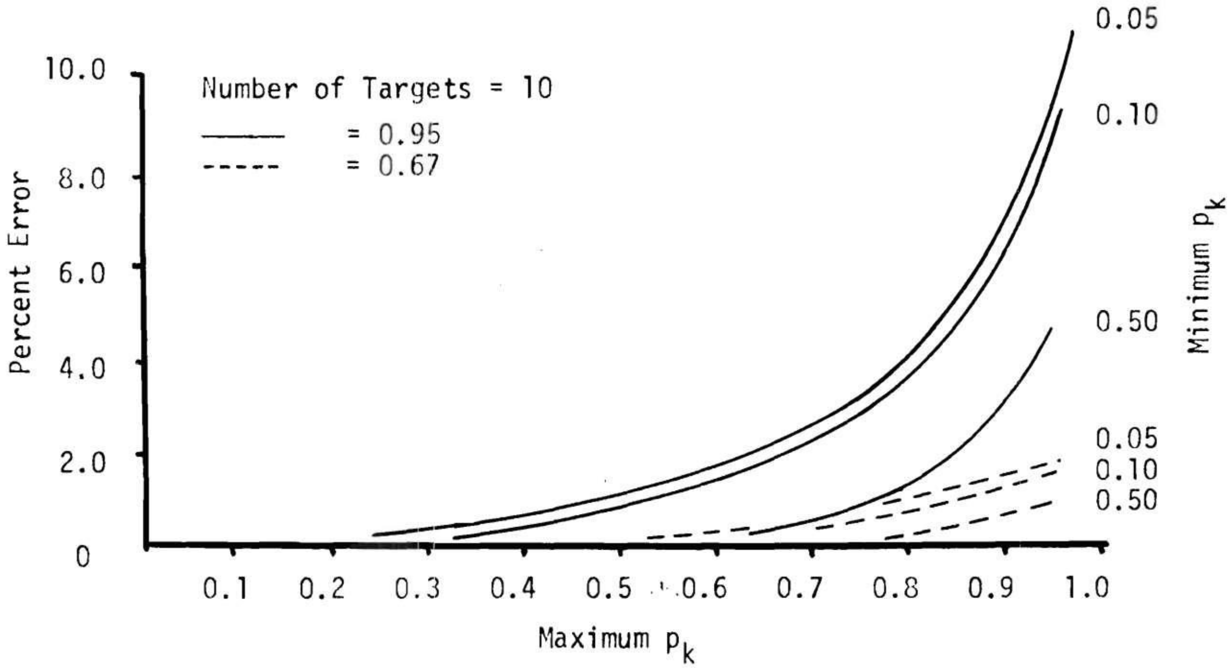

Figure 4. Maximum Infeasibility in Lower Bound Linear Approximation versus Minimum and Maximum p_k for = 0.95 and 0.67.

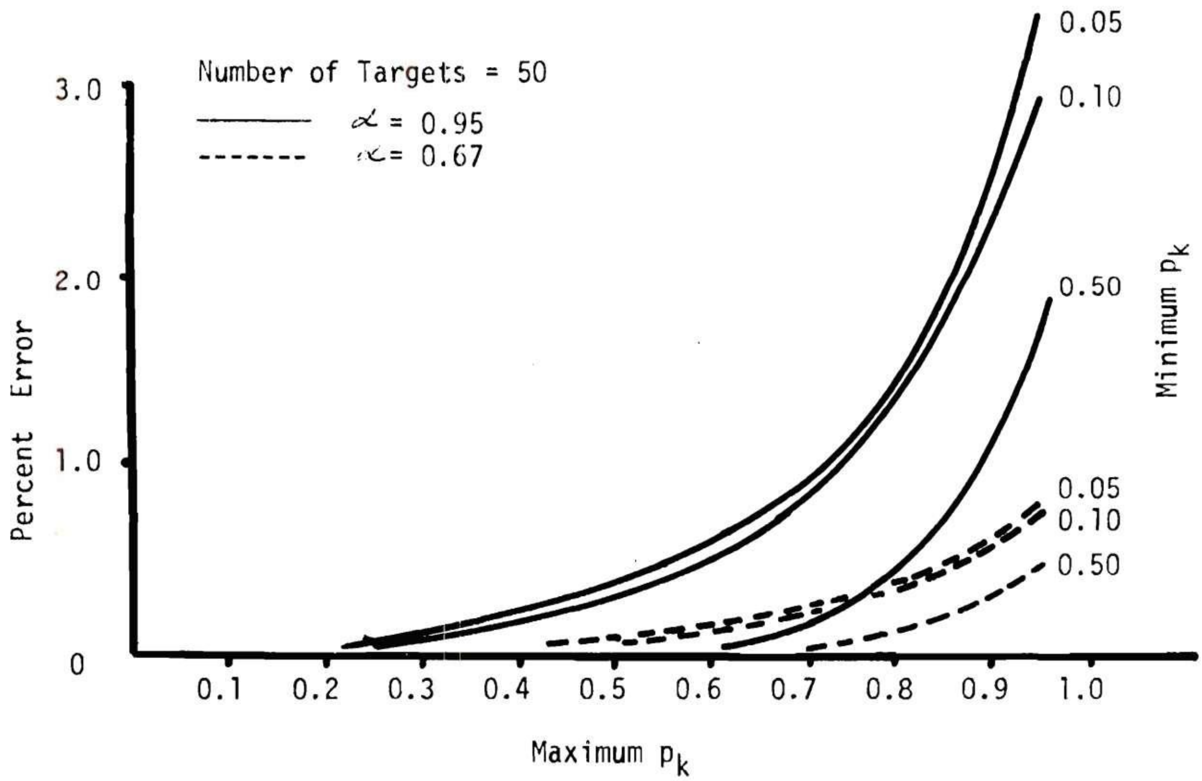

Figure 5. Maximum Infeasibility in Lower Bound Linear Approximation versus Minimum and Maximum p_k for $\alpha = 0.95$ and 0.67.

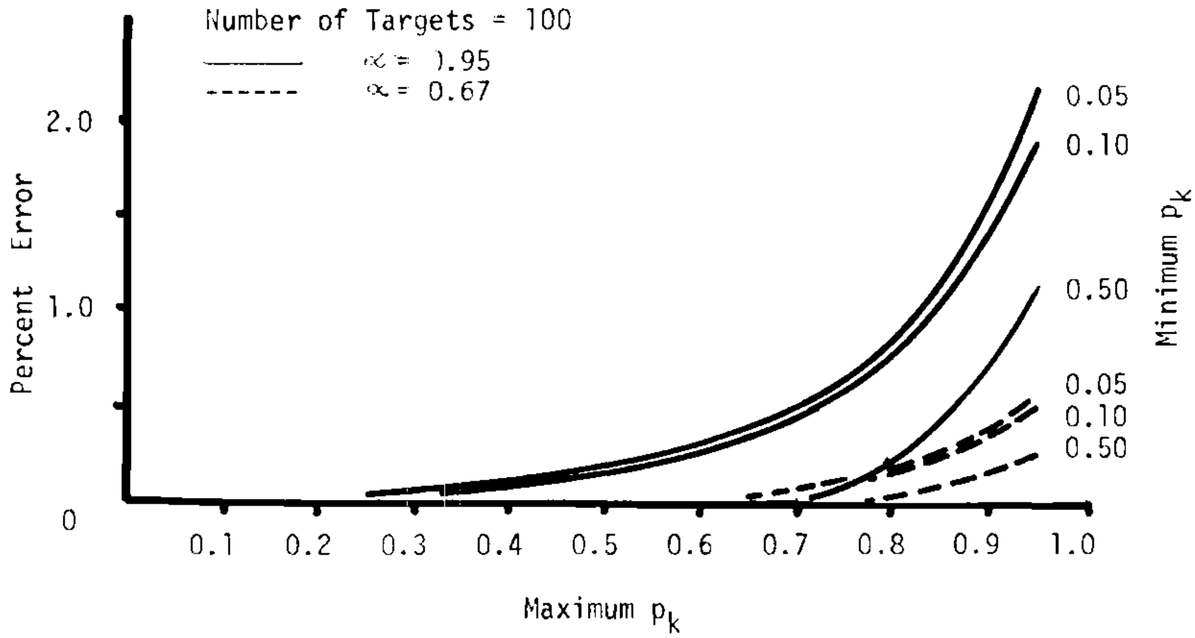

Figure 6. Maximum Infeasibility in Lower Bound Linear Approximation versus Minimum and Maximum p_k for $\alpha = 0.95$ and 0.67.

CHAPTER VI

SUMMARY AND COMMENTS

In this thesis the problem of the selection of a minimum cost inventory of conventional weaponry which will satisfy contingency requirements with a high degree of certainty has been solved. The problem has been put into a linear programming format by two approximations: first, use of the normal approximation to the binomial distribution has made possible the development of a deterministic mathematical statement which, if satisfied, implies that probabilistic certain constraints imposed on the inventory are satisfied; secondly, the above deterministic statements, called certainty equivalents, may be approximated with linear constraints involving minimal error. In both cases the errors resulting from the approximations are shown to be very small.

Two linear approximations to the certainty equivalents are developed. The first is a lower, support hyperplane to the convex certainty equivalent, and the second is an upper support hyperplane. Replacement of the certainty equivalents with the lower support planes and solution of the linear program results in an inventory which may be slightly infeasible to the certainty equivalent constraints. Replacement with the upper support planes gives a solution which, though feasible, results in more than minimum satisfaction of the certainty equivalents, or, "overkill." Solving both problems gives an upper and lower bound on the total, optimal inventory cost. It was also shown that a convex combination of the two approximate solutions may be formed that results in a feasible improvement on the upper bound problem solution.

Little or no discussion has been devoted to the actual solution of the linear program resulting from the approximations and this is as intended. It is expected that standard linear programming techniques will be applied. Normally, as seen in the expression of the problems in (1) and (2) of Chapter V, the linear programs that must be solved are decomposable, thus adding to the ease of solution of a large problem.

Tne depth and uniqueness of this thesis material is certainly not in the final solution of a linear program. Rather it is in the formulation of the problem in a manner such that it may be solved by this readily available technique, and the validation of the formulation contained in the chapter devoted to sensitivity analysis of the approximations used. Indeed, it is this ease of solution which enhances the probability that this approach will be accepted by analysts addressing real problems.

Since the work on this thesis has begun, considerable interest has been generated in the offices of USAF Headquarters where the inventory problem is now being worked. It is expected that this technique will be put to use as soon as existing data can be put into the proper format. Certainly, though, use will not be solely restricted to complex problems as large as the one outlined in Chapter I. Many other smaller problems may be addressed in this manner in the author's present location, the Air Force Armament Laboratory, at Eglin AFB, Fla.

Finally, the solution of the problem contains much more information than merely the optimal force mix. Through the definition of what is referred to as a weapon, the problem solution also contains an indication of what tactics are to be expected to be used, as what weather conditions these tactics will be used under. This has obvious implications on USAF combat pilot training. Additionally, standard linear programming sensitivity analysis yields much information about the implications of enlarging delivery aircraft force size, effect of variation of weapon unit cost, and so on.

In conclusion, it should be clear that the formulation developed herein is extremely useful to the R and D analyst in that it affords him the capability to look at the conventional weapons inventory as a system, and measure the response of that system to his concepts and projected developments.

APPENDIX I

In Chapter IV it was stated that upper bound linear approximations to the certainty equivalents may be developed and used in assessing the accuracy of the lower bound hyperplane approximation. This upper bound will be a plane parallel to the lower bound hyperplane and tangent to the certainty equivalent at some point $\hat{\underline{x}}$. The problem of finding the mathematical expression for this plane is, in itself, an optimization problem and is restated here for reference:

$$\text{maximize } \underline{hx} \tag{A.1}$$

$$\text{subject to } g(\underline{x}) \leq 0; \quad x \geq 0$$

where the i subscript of Chapter IV has been dropped for notational convenience.

The problem in (A.1) has a solution if the feasible set is closed and bounded. Define the set of feasible points to (A.1) as $S = \left\{\underline{x} | g(\underline{x}) \leq 0; \ \underline{x} \geq 0\right\}$. Obviously S is closed. It will now be shown that S is bounded.

$g(\underline{x})$ was derived as

$$g(\underline{x}) = \underline{mx} - U(\underline{vx})^{1/2} - H \leq 0 \tag{A.2}$$

or

$$\underline{mx} \leq U(\underline{vx})^{1/2} + H \tag{A.2}$$

By the Schwartz inequality

$$\| \underline{vx} \| \leq \|\underline{v}\| \, \|\underline{x}\| \tag{A.3}$$

and hence

$$\| \underline{vx} \|^{1/2} \leq \| \underline{v} \|^{1/2} \| \underline{x} \|^{1/2} \qquad \text{(A.4)}$$

Substituting (A.4) into (A.2) yields

$$\underline{mx} \leq U \, \| \underline{v} \|^{1/2} \, \| \underline{x} \|^{1/2} + H \qquad \text{(A.5)}$$

Now $\underline{mx} = (m_1 x_1 + m_2 x_2 + \ldots + m_W x_W)$, so let $\tilde{m} = \min_j m_j > 0$. Since $m_j = p_j > 0$, then $\tilde{m} > 0$. $\underline{mx}$ may be rewritten as

$$\underline{mx} = \tilde{m} \left(\frac{m_1}{\tilde{m}} x_1 + \frac{m_2}{\tilde{m}} x_2 + \ldots + \frac{m_W}{\tilde{m}} x_W \right)$$

Note that $\frac{m_j}{\tilde{m}} \geq 1$. Therefore $\underline{mx} \geq \tilde{m}(x_1 + x_2 + \ldots + x_W)$ since $x_i \geq 0$. Because $x_i \geq 0$, $x_1 + x_2 + \ldots + x_W \geq \| \underline{x} \|$. Then $\underline{mx} \geq \tilde{m} \| \underline{x} \|$, and

$$\underline{\tilde{m}} \| \underline{x} \| \leq \underline{mx} \leq U \| \underline{v} \|^{1/2} \, \| \underline{x} \|^{1/2} + H \qquad \text{(A.6)}$$

Consider points within S such that $\| \underline{x} \| \geq 1$. (If there are no such points, then clearly S is bounded.) Divide (A.6) by $\| \underline{x} \|^{1/2}$; then

$$\underline{\tilde{m}} \| \underline{x} \|^{1/2} \leq U \| \underline{v} \|^{1/2} + \frac{H}{\| \underline{x} \|^{1/2}}$$

For $|\underline{x}| \geq 1$, $1/|\underline{x}| \leq 1$, and also

$$1/\| \underline{x} \|^{1/2} \leq 1 \qquad \text{(A.7)}$$

Therefore $H/\| \underline{x} \|^{1/2} \leq H$ and (A.6) becomes

$$\underline{\tilde{m}} \| \underline{x} \|^{1/2} \leq U \| \underline{v} \|^{1/2} + H = k \qquad \text{(A.8)}$$

Thus, by (A.8),

$$\|\underline{x}\|^{1/2} \leq \frac{k}{\tilde{m}} = L$$

and L is simply a positive constant. So $\underline{x} \in X$ implies that $\|\underline{x}\| \leq \max (L^2,1)$, therefore S is bounded and (A.1) has a solution.

Addressing the problem of (A.1) directly, the Kuhn-Tucker conditions may be written

$$h_j - \lambda \frac{\partial g(\hat{\underline{x}})}{\partial x_j} \leq 0 \quad j = 1,2,... \tag{A.9}$$

$$\hat{x}_j[h_j - \lambda \frac{\partial g(\hat{x})}{\partial x_j}] = 0 \quad j = 1,2,... \tag{A.10}$$

$$\hat{\underline{x}} \geq 0 \quad j = 1,2,... \tag{A.11}$$

$$g(\hat{\underline{x}}) \leq 0 \tag{A.12}$$

$$\lambda g(\hat{\underline{x}}) = 0 \tag{A.13}$$

$$\lambda \geq 0 \tag{A.14}$$

Suppose $\hat{x}_i$ and $\hat{x}_j$ are two components of the optimum solution $\underline{x}$ that are positive. Then (A.10) implies that

$$h_j - \lambda \frac{\partial g(\hat{\underline{x}})}{\partial x_j} = h_i - \lambda \frac{\partial g(\hat{\underline{x}})}{\partial x_i} = 0. \tag{A.15}$$

Now λ must be unique so

$$\lambda = \frac{h_i}{\partial g(\hat{\underline{x}})/\partial x_i} = \frac{h_j}{\partial g(\hat{\underline{x}})/\partial x_j} \tag{A.16}$$

The partial derivative

$$\frac{\partial g(\hat{\underline{x}})}{\partial x_k} = m_k - \frac{Uv_k}{2} (\underline{v}\,\hat{\underline{x}})^{-1/2} \qquad (A.17)$$

and $\underline{v\hat{x}} = \hat{V}$ is a scalar constant. Substituting the expressions for the partial derivatives in (A.16) and solving for $\hat{V}$, it is found that

$$\hat{V}^{1/2} = \frac{U}{2} \frac{\alpha_i y_i - \alpha_j y_j}{y_i - y_j} \qquad (A.18)$$

where

$$y_k = y(\alpha_k) = m_k/h_k = H + \frac{U^2\alpha_k}{2} + \frac{U}{2}[4H\alpha_k + (U\alpha_k)^2]^{1/2};k=i,j. \quad (A.19)$$

From (A.15) $\lambda = h_i/[\partial g(\hat{\underline{x}})/\partial x_i] = 0$. Substituting the appropriate expression for the partial derivative this becomes

$$\lambda = \frac{h_i}{m_i - \frac{Uv_i}{2} (\hat{V})^{-1/2}} \qquad (A.20)$$

Using (A.18) in (A.20) an expression for λ may be obtained through some algebraic manipulation:

$$\lambda = \frac{\alpha_i y_i - \alpha_j y_j}{y_i y_j(\alpha_i - a_j)} \qquad (A.21)$$

Recall that in Chapter IV it was stated that no two weapons have the same α values (because no two weapons have the same kill probabilities). Then it follows that λ exists and further, since $\alpha_i > \alpha_j$ implies $y_i > y_j$, or vice-versa, then $\lambda > 0$.

To this point it has been assumed that at least two components of $\hat{\underline{x}}$ are positive. By way of forming a contradiction, assume three components of $\hat{\underline{x}}$, say $\hat{x}_i$, $\hat{x}_j$, and $\hat{x}_k$ are positive. For the pair $\hat{x}_i$ and $\hat{x}_k$ (as well as the pair $\hat{x}_j$ and $\hat{x}_k$) (A.9) must be satisfied at equality. For both the above pairs an expression for $\hat{V}^{1/2}$ is obtained, identical in form to (A.18). Note these two expressions respectively as $\hat{V}_{ik}^{1/2}$ and $\hat{V}_{jk}^{1/2}$. Remembering the definition $\hat{V}^{1/2} = \underline{v}\hat{\underline{x}}$, it is obvious that $\hat{V}_{ik}^{1/2}$ must be equal to $\hat{V}_{jk}^{1/2}$.

Digressing momentarily, investigation of the second derivative of $y(\alpha)$ shows that this function is strictly concave for $\alpha > 0$. Further, calculation of the second derivative of $1/y(\alpha)$ shows that, if $y(\alpha)$ is strictly concave over $\alpha > 0$, then $1/y(\alpha)$ is strictly convex over that set. In other words, $1/y(\alpha)$ is a strictly convex function for $\alpha > 0$. Since no two components of $\underline{x}$ have the same α value, with no loss in generality it may be assumed that $\alpha_i > \alpha_j > \alpha_k$. Rewriting α_j as $\alpha_j = \gamma\alpha_i + (1-\gamma)\alpha_j$, for $\gamma \varepsilon (0,1)$, strict convexity of $1/y(\alpha)$ means that

$$\frac{1}{y_j} < \frac{\gamma}{y_i} + \frac{(1-\gamma)}{y_k} \tag{A.22}$$

again using the notation that $y(\alpha_i) = y_i$.

Algebraic manipulation of (A.22), using the fact that α_j may be expressed as $\gamma\alpha_i + (1-\gamma)\alpha_k$, yields

$$\frac{u}{2}\left[\frac{\alpha_i y_i - \alpha_k y_k}{y_i - y_k}\right] > \frac{u}{2}\left[\frac{\alpha_j y_j - \alpha_k y_k}{y_j - y_k}\right] \tag{A.23}$$

But the left and right hand sides of (A.23) are respectively $\hat{V}_{ik}^{1/2}$ and $\hat{V}_{jk}^{1/2}$ which must be equal. Thus, no more than two components of $\hat{\underline{x}}$ may be positive.

If $\lambda = 0$ in the optimal solution, then, to satisfy (A.10), all $\hat{x}_j h_j$ must be zero. But, by the structure of the problem, all $h_j > 0$. So, if $\lambda = 0$ all $\hat{x}_j = 0$ and the optimal solution to (A.1) is $\hat{\underline{x}} = \underline{0}$. But this solution cannot be optimal because at $\hat{\underline{x}} = \underline{0}$ an $\varepsilon > 0$ may be found such that letting the j^{th} component, $\hat{x}_j = \varepsilon$ results in a new point which both is feasible and improves the objective function $\underline{hx}$. Therefore, $\lambda > 0$ in the optimal solution to (A.1).

Since $\lambda > 0$ at optimality, at this point (A.12) must be solved at equality. Suppose only one component of $\hat{\underline{x}}$, say $\hat{x}_j$, is positive. Then, by a previous calculation, the value of x_j that makes $g(\underline{x}) = 0$ is X_j and the optimal solution is $\underline{X}_j$. However, by the construction of the lower bound hyperplane, if $\underline{hx}$ contains the point $\underline{X}_j$, it also contains all other points $\underline{X}_k$, $k = 1, 2, \ldots, N$. This means that if $x_j = X_j$, then $x_k = X_k$, for all k. Therefore $\hat{\underline{x}}$ must have more than one positive component. It follows, then, since $\hat{\underline{x}}$ cannot have more than two positive components, that exactly two components of $\hat{\underline{x}}$ are positive.

Let u be the index such that $\alpha_u = \max_j \alpha_j$. Likewise, let ℓ be such that $\alpha_\ell = \min_j \alpha_j$.

Suppose, for $\hat{x}_\ell$, (A.9) is satisfied by

$$h_\ell - \lambda \frac{\partial g(\hat{\underline{x}})}{\partial x_\ell} = h_\ell - \lambda \left[m_\ell - \frac{Uv_\ell}{2} (\underline{v}\,\underline{x})^{-1/2}\right] < 0 \qquad (A.25)$$

This implies that $\hat{x}_\ell = 0$ and that two other components, say $\hat{x}_i$

and $\hat{x}_k$, i, k $\neq \ell$ must be positive. Recall that, if $\hat{x}_i > 0$ and $\hat{x}_k > 0$, $\hat{V}^{1/2}$ is found to be

$$\hat{V}^{1/2} = \frac{U}{2}\left[\frac{\alpha_i y_i - \alpha_k y_k}{y_i - y_k}\right] \tag{A.26}$$

and λ is found to be

$$\lambda = \frac{y_i\alpha_i - y_k\alpha_k}{y_i y_k(\alpha_i - \alpha_k)} \tag{A.27}$$

Since no two components of $\hat{x}$ may have the same α's, arbitrarily, let $\alpha_i > \alpha_k$. By definition $\alpha_i > \alpha_\ell$, so α_k may be written $\alpha_k = \gamma\alpha_i + (1-)\alpha_\ell$, where $\gamma \varepsilon (0,1)$. Using this, together with (A.26) and (A.27) in (A.25), it can be shown that (A.25) implies

$$\frac{1}{y(\alpha_k)} > \frac{\gamma}{y(\alpha_i)} + \frac{(1-\gamma)}{y(\alpha_\ell)} \tag{A.28}$$

But this cannot be true since $1/y(\alpha)$ is strictly convex. Thus, (A.9) must be satisfied at equality for i = ℓ.

A similar assumption to (A.25), for an arbitrary member of the set u, leads to the result

$$\frac{1}{y(\alpha_i)} > \frac{\gamma}{y(\alpha_u)} + \frac{(1-\gamma)}{y(\alpha_k)} \quad \text{where } \alpha_i = \gamma\alpha_u + (1-\gamma)\alpha_k$$

which, again, is not true by the convexity of $1/y(\alpha)$.

It follows, then, that (A.9) must be satisfied at equality for components u and ℓ. Then $V^{1/2}$ and λ may be written respectively

$$V^{1/2} = \frac{U}{2}\left[\frac{\alpha_u y_u - \alpha_\ell y_\ell}{y_u - y_\ell}\right] \tag{A.29}$$

and

$$\lambda = \frac{\alpha_u y_u - \alpha_\ell y_\ell}{y_u y_\ell(\alpha_u - \alpha_\ell)} \tag{A.30}$$

Consider some component i such that $i \neq u$ or ℓ. Then α_i may be written $\alpha_i = \gamma\alpha_u + (1 - \gamma)\alpha_\ell$ and

$$\frac{1}{y(\alpha_i)} > \frac{\gamma}{y(\alpha_u)} + \frac{(1-\gamma)}{y(\alpha_\ell)} \tag{A.31}$$

After considerable algebraic manipulation, using (A.29) and (A.30), (A.31) is seen to imply

$$h_i - \lambda[m_i - \frac{Uv_i}{2}(\hat{V})^{-1/2}] = h_i - \lambda\frac{\partial g(\hat{\underline{x}})}{\partial x_i} < 0 \tag{A.32}$$

But this is just the Kuhn-Tucker condition (A.9) for $\hat{x}_i$. Thus, to satisfy (A.9), all $\hat{x}_i$, such that $i \neq u$ or ℓ, must be zero.

Summarizing to this point, it has been shown that exactly two components of $\hat{\underline{x}}$ are positive. Of these two components, one corresponds to the weapon having the maximum α_j (minimum p_j) and the other to the weapon having the minimum α_j (maximum p_j).

The point at which $\underline{hx}$ obtains its optimal value is calculated by making the following observations.

At $\hat{\underline{x}} = (0,\ldots,0,\hat{x}_\ell,0,\ldots,0,\hat{x}_u,0,\ldots,0)$ let

$$\underline{m\hat{x}} = m_u\hat{x}_u + m_\ell\hat{x}_\ell = \hat{M}$$

and

$$\underline{v\hat{x}} = v_u\hat{x}_u + v_\ell\hat{x}_\ell = \hat{V}$$

where $\hat{x}_u$ and $\hat{x}_\ell$ correspond to the maximum and minimum α_j respectively.

Rewriting $g(\hat{x})$ in terms of $\hat{M}$ and $\hat{V}$:

$$g(\hat{\underline{x}}) = \hat{M} - U\hat{V}^{1/2} - H$$

and, since at $\hat{\underline{x}}$, $g(\hat{\underline{x}}) = 0$,

$$\hat{M} - U\hat{V}^{1/2} - H = 0$$

or
$$\hat{M} = m_u\hat{x}_u + m_\ell\hat{x}_\ell = U\hat{V}^{1/2} + H \tag{A.33}$$

The value for $\hat{V}$ is obtained by squaring (A.29), so

$$v_u\hat{x}_u + v_\ell\hat{x}_\ell = \hat{V} = \frac{U^2}{4}\left[\frac{\alpha_u y_u - \alpha_\ell y_\ell}{y_u - y_\ell}\right]^2 \tag{A.34}$$

(A.33) and (A.34) may be solved simultaneously to yield values for x_u and x_ℓ. These are

$$\hat{x}_u = \frac{1}{m_u} \cdot \frac{\hat{V} - \alpha_\ell\hat{M}}{\alpha_u - \alpha_\ell} \tag{A.35}$$

$$\hat{x}_\ell = \frac{1}{m_\ell} \cdot \frac{\alpha_u\hat{M} - \hat{V}}{\alpha_u - \alpha_\ell} \tag{A.36}$$

Summarizing, it was first shown that (A.1) has a solution as given above. The strict convexity of $1/y(\alpha)$ was seen to be a powerful tool in proving that $\hat{\underline{x}}$ has exactly two positive components and that they correspond to the maximum and minimum α_j (or, respectively, the minimum and maximum p_j). $\hat{\underline{x}}$ may be computed through the use of (A.35) and (A.36).

APPENDIX II

For the purpose of clarifying ideas presented in this thesis a small numerical example is given in this appendix. Three weapons are used against two targets. It is assumed that the cost of using these weapons does not vary from target to target, though, of course, the destruction probability does. Pertinent data are presented in Table 3 below:

Table 3. Example Problem Data

	p_{1j}	p_{2j}	c_j
Weapon 1	0.1	0.2	1
Weapon 2	0.4	0.6	5
Weapon 3	0.9	0.3	10
H_i	50	100	
t_{α_i}	-1.645	-1.645	

Fifty of target type 1 must be destroyed, with a certainty of 0.95, while 100 of type 2 must be killed with the same certainty. The certainty equivalents for the two targets are written below, in the form of (4) of Chapter IV:

Target 1: $0.1x_1 + 0.4x_2 + 0.9x_3 - (1.645)(0.09x_1 + 0.24x_2 + 0.09x_3)^{1/2} - 50 \geq 0$

Target 2: $0.2x_4 + 0.6x_5 + 0.3x_6 - (1.645)(0.16x_4 + 0.24x_5 + 0.21x_6)^{1/2} - 100 \geq 0$

or, in the form of (9) of Chapter IV:

Target 1: $(0.1x_1 + 0.4x_2 + 0.9x_3)^2 - (10.244x_1 + 40.649x_2 + 90.244x_3)$

$$- 2500 \geq 0$$

Target 2: $(0.2x_4 + 0.6x_5 + 0.3x_6)^2 - (40.433x_4 + 120.649x_5 + 60.568x_6)$

$$- 10000 \geq 0$$

The table below yields the data necessary for development of the upper and lower hyperplanes:

Table 4. Data for Construction of Upper and Lower Bound Hyperplanes

	Target 1	Target 2		Target 1	Target 2
α_u	0.9	0.8	h_u	0.001604	0.001727
α_l	0.1	0.4	h_l	0.01672	0.005408
m_u	0.1	0.2	y_u	62.344	115.808
m_l	0.9	0.6	y_l	53.828	110.947
v_u	0.09	0.16	$V^{1/2}$	4.384	8.167
v_l	0.09	0.24	V	19.244	66.701
			M	57.212	113.435

The lower bound hyperplanes are:

Target 1: $1.604x_1 + 6.865x_2 + 16.720x_3 \geq 1000$

$$1.727x_4 + 5.408x_5 + 2.165x_6 \geq 1000$$

Using equations (12) and (13) of Chapter IV the points of tangency of the upper bound hyperplanes to the certainty equivalents are:

Target 1: $(x_u, 0, x_l) = (170.111, 0, 43.489)$

Target 2: $(x_u, x_l, 0) = (267.788, 95.564, 0)$

The maximum infeasibility in the Target 1 certainty equivalent, solving the lower bound problem, is about 1.05 targets, or about 2.1 percent; for Target 2 it is about 2.4 targets (2.4 percent). The right hand sides for the upper bound hyperplanes may be calculated from the data of Table 4 and are:

Target 1: $\underline{h}_1\underline{x}_1 = \delta_1 = 1020.04$

Target 2: $\underline{h}_2\underline{x}_2 = \delta_2 = 1002.26$

In the interest of solving an example problem let us impose a few operational and equipment constraints. First, assume that the total number of missions available is 2000. Thus

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \leq 2000$$

Additionally, assume that special equipment restrictions limit the missions even further:

$$\begin{aligned}
x_1 \qquad + x_4 \qquad\qquad &\leq 200\\
x_2 \qquad + x_5 \qquad &\leq 50\\
x_3 \qquad + x_6 &\leq 400\\
x_3 + x_4 + x_5 + x_6 &\leq 457\\
-.259x_4 + x_5 \qquad &\leq 169
\end{aligned}$$

Putting these constraints together with the lower bound hyperplanes and solving the linear program ((1) of Chapter V), the solution below results:

$$\underline{x}_L^* = (0,\ 0,\ 59.809,\ 200,\ 50,\ 146.922)$$

This solution is feasible to the certainty equivalent for Target 1 and has an infeasibility of 0.156 for Target 2. Note, of course, that, since fractions of missions cannot be flown, the L.P. solution would be rounded up to the next whole integer and then the solution would be feasible. Since the certainty equivalents are so nearly linear, in most cases this is very likely the optimum integer solution also.

Replacement of lower bound approximations with upper bound hyperplanes results in the L.P. solution:

$$\underline{x}_U^* = (0,\ 0,\ 60{,}634,\ 196.111,\ 50,\ 150.255)$$

and this solution is feasible to both constraints. (In fact, where the requirement is that $g_i(\underline{x}_i) \geq 0$, $g_1(\underline{x}_{1U}^*) = 0.728$ and $g_2(\underline{x}_{2U}^*) = 0.059$.)

The comparative values for $z_U = \underline{cx}_U^*$ and $z_L = \underline{cx}_L^*$ are:

$$z_U = 2555.0 > z_L = 2517.31$$

Applying the improvement technique of Chapter IV to this example

$$\underline{x}_c = [\gamma \underline{x}_U^* + (1-\gamma) x_L^*] = [0, 0, (59.809 + 0.825\gamma), (200.0 - 3.889\gamma), 50.0, (146.992 + 3.263\gamma)]$$

$g_1(\underline{x}_c)$ results in the quadratic constraint

$$0.552\gamma^2 + 3.537\gamma + 0.506 \geq 0$$

and $g_1 \geq 0$ for $\gamma \geq 0$ and for $\gamma \leq -10.021$. Likewise, the constraint $g_2(x_c)$ results in the expression

$$0.404\gamma^2 + 5.501\gamma - 3.707 \geq 0$$

which implies $g_2 \geq 0$ for $\gamma \geq 0.6707$ and for $\gamma \leq -136.834$.

The problem as formed in (18) of Chapter IV is

Minimize $\underline{c}\,\underline{x}_c = 2517.31 + 38.69\gamma$

Subject to: either $\gamma \geq 0$ or $\gamma \leq -10.021$

either $\gamma \geq 0.6707$ or $\gamma \leq -136.834$

$$0 \leq \gamma \leq 1$$

This problem reduces to simply

Minimize γ

Subject to: $0.6707 \leq \gamma \leq 1.0$

and the obvious answer is $\gamma = 0.6707$, generating

$$\underline{x}_c = (0,\ 0,\ 60.362,\ 197.392,\ 50.0,\ 149.180)$$

Note that $z_c = \underline{cx}_c = 2542.81$ and $z_U^* > z_c^* > z_L^*$.

In summary all solutions are close, with a disparity of only about 1.8 percent in their total costs. This is in spite of the fact that, in the weapons associated with target 1, there was a rather large kill probability of 0.9.

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